Asymptotics for first-passage times of Lévy processes and random walks

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Abstract

We study the exact asymptotics for the distribution of the first time τ_x a Lévy process X_t crosses a negative level $-x$. We prove that $\mathbf{P}(\tau_x > t) \sim V(x)\mathbf{P}(X_t \geq 0)/t$ as $t \to \infty$ for a certain function $V(x)$. Using known results for the large deviations of random walks we obtain asymptotics for $\mathbf{P}(\tau_x > t)$ explicitly in both light and heavy tailed cases. We also apply our results to find asymptotics for the distribution of the busy period in an M/G/1 queue.

Keywords: Lévy processes, random walks, busy period, first-passage times, subexponential distributions, large deviations, single server queue.

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Introduction

Let $\{X_t\}_{t \geq 0}$ be a Lévy process with the characteristic function $\mathsf{E}\{e^{i\theta X_t}\} = e^{t\Psi(\theta)}, t \geq 0$, where Ψ is given by the Lévy-Khinchine formula[33]

$$\Psi(\theta) = iA\theta - \frac{1}{2}\sigma^2\theta^2 + \int_{-\infty}^{+\infty}(e^{i\theta x} - 1 - i\theta x \mathbf{1}_{[-1,1]})\Pi(\mathrm{dx}). \tag{1}$$

For $x \geq 0$ let

$$\tau_x = \min\{t \geq 0 : X_t < -x\}$$

be the first-passage time. Throughout we assume that the Lévy process X_t drifts to $-\infty$ a.s. Rogozin's criterion [31] (see also [5, p. 167] or [33, theorem 48.1]) says that $X_t \to -\infty$ if and only if

$$\int_1^\infty t^{-1}\mathsf{P}\{X_t \geq 0\}\mathrm{dt} < \infty. \tag{2}$$

This assumption implies that τ_x is a proper random variable with finite expectation:

$$\tau_x < \infty, \quad \mathsf{E}\{\tau_x\} < \infty.$$

The aim of this work is to study asymptotics

$$\mathsf{P}\{\tau_x > t\}$$

when $x > 0$ is fixed and $t \to \infty$.

An analogue of the Lévy processes for discrete time is random walks. Let $S_n = \xi_1 + \cdots + \xi_n$ be a random walk with i.i.d. increments and assume that $S_n \to -\infty$ a.s. Then an analogue of τ_x is the stopping time $\nu_x = \min\{n \geq 1 : S_n < -x\}$. Asymptotics for ν_x have been studied in [16] for $x = 0$, and in [13], [6] for $x > 0$. In [16], it is shown that $\mathsf{P}\{\nu_0 > n\} \sim \mathsf{P}\{S_n \geq 0\}/n$ if the latter is a subexponential sequence (see definition below). In [13] and [6], the asymptotics for ν_x have been found when $x > 0$. In these papers the authors considered separately several classes of distribution of ξ: regularly varying distributions, distributions satisfying Cramér's condition and light-tailed distributions which do not satisfy

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Cramér's condition. For each of these classes, they show that $\mathsf{P}\{\nu_x > n\}$ is asymptotically proportional to $\mathsf{P}\{S_n \geq 0\}/n$, using large-deviations asymptotics for $\mathsf{P}\{S_n \geq 0\}$. In [10] (see also [9]) it has been shown that the same asymptotic equivalence holds for the so-called semi-exponential distributions with parameter smaller that $1/2$ (see definition in [9]). In [24] the latter result has been generalised to the case of semi-exponential distributions with parameter smaller than 1. We note also that in [24] x may depend on n. We also mention the paper [8] where the same problem was studied under Cramér's assumptions and in the case when x may depend on n.

For Lévy processes, the asymptotics for τ_x have been obtained in [22] for the light-tailed distributions. Necessary and sufficient conditions for the existence of moments of τ_x have been obtained in [14].

In our work we develop the approach proposed in [13], [6]. The main results of our paper are Theorems 1.1 and 1.2. Theorem 1.1 states that under some natural assumptions,

$$\mathsf{P}\{\tau_x > t\} \quad \sim \quad V(x)\frac{\mathsf{P}\{X_t \geq 0\}}{t},$$

for some function $V(x)$ depending only on x. Theorem 1.2 shows that under identical conditions for both Lévy processes and random walks,

$$\mathsf{P}\{\nu_x > n\} \quad \sim \quad V_{rw}(x)\frac{\mathsf{P}\{S_n \geq 0\}}{n},$$

$$\mathsf{P}\{\tau_x > t\} \quad \sim \quad V(x)e^{-\gamma(t-[t])}\frac{\mathsf{P}\{X_{[t]} \geq 0\}}{t}$$

for some $\gamma \geq 0$, where by $[t]$ we denote the integer part of t (the largest integer smaller than t) and functions $V(x)$ and $V_{rw}(x)$ depend only on x. It is worth mentioning that for the case of Lévy processes both conditions of Theorem 1.2 and its result are given in terms of the values of the process at discrete times. Therefore, the problem of finding asymptotics for $\mathsf{P}\{\tau_x > t\}$ and $\mathsf{P}\{\nu_x > n\}$ is reduced to finding asymptotics for $\mathsf{P}\{S_n \geq 0\}$, or $\mathsf{P}\{\tilde{S}_n \geq na\}$ where $a = -\mathsf{E}\{\xi_1\}$ and $\tilde{S}_n = S_n + na$ is a random walk with zero drift. This is a problem of large deviations of sums of independent identically distributed random variables which is extensively studied in literature. We apply known results to obtain explicit asymptotics in various cases. It appears that in all cases when asymptotics for $\mathsf{P}\{S_n > 0\}$ can be found explicitly, the conditions of Theorem 1.2 are satisfied and hence, asymptotics for the tail distribution of τ_x and ν_x can also be found explicitly.

We consider distributions with heavy tails (such that $\mathsf{E}\{e^{\varepsilon\xi_1}\} = \infty$ for all $\varepsilon > 0$) and distributions with light tails (for which the latter condition is not fulfilled) separately. Moreover, it has been pointed out by various authors that for the problem of large deviations of sums of heavy-tailed random variables, one should indicate two classes of distributions: those with tails lighter and heavier than $e^{-\sqrt{t}}$ (we say that the tail of a distribution F is lighter than a function f if $\overline{F}(t)/f(t) \to 0$ as $t \to \infty$ and heavier than a function f if $\overline{F}(t)/f(t) \to \infty$). For the case of heavy-tailed distributions some new results on large deviations are presented in [12]. We also state some of these results in the present paper, however, we do not concentrate on the problem of large deviations here.

Theorem 1.2 can also be applied to the case of light-tailed distributions under some further assumptions. In particular, the conditions of Theorem 1.2 are fulfilled if the distribution of ξ_1 satisfies the so-called Cramér's (or classical) conditions. With the help of Theorem 1.2 we can also cover the so-called intermediate case, when ξ_1 has a distribution with a light tail but does not satisfy Cramér's condition.

Another motivation for our work was to find asymptotics for the busy period in a stable $M/G/1$ queue. Let $A_1, A_2, \ldots$ and $B_1, B_2, \ldots$ be two mutually independent sequences each consisting of independent and identically distributed random variables. Assume that $\{A_i\}$ are inter-arrival times and $\{B_i\}$ are service times. We assume throughout that $\mathsf{E}\{B_1\}/\mathsf{E}\{A_1\} = \rho < 1$ so that the system is stable. We use the common notation: we denote by $M/G/1$ the system when A_i are exponential random variables; in the

case of a general i.i.d. sequence $\{A_i\}$, we denote the system by $GI/GI/1$. Denote $N(t) = \max\{n : A_1 + \ldots + A_n \leq t\}$. Put $X_0 = 0$ and

$$X_t = \sum_{i=1}^{N(t)} B_i - t. \tag{3}$$

Then the busy period of the system with initial work $x > 0$ may be defined as

$$bp(x) = \inf\{t : X_t < -x\}.$$

Hence, in an $M/G/1$ queue, finding asymptotics for the tail of $bp(x)$ is equivalent to finding asymptotics for the tail of τ_x when X_t is a compound Poisson process without negative jumps.

The tail behavior of the busy period in these systems has been studied by various authors under different assumptions. Under Cramér-type assumptions, asymptotics for the $M/G/1$ setting were studied in [1] and for the $GI/G/1$ setting — in [28]. Most of the papers on the tail behaviour of the busy period are devoted to studying the case when B_1 has a subexponential distribution. All these papers investigate the asymptotic behaviour of bp — busy period of the queue under the condition that the first customer arriving at the system finds it empty. In [23], it was shown that if B_1 has a regularly varying distribution then

$$\mathsf{P}\{bp > t\} \sim \mathsf{E}\{\nu_0\}\mathsf{P}\{B_1 > (1-\rho)t\} \tag{4}$$

as $t \to \infty$. This result has been generalised in [34] to the case of a $GI/G/1$ queue and under the assumption that the tail $\overline{B}(t) = \mathsf{P}\{B_1 > t\}$ satisfies an extended regular variation condition (see [7]).

Later on, it has been shown in [4] and [19], that the asymptotics (4) hold for the $GI/G/1$ model for another subclass of heavy-tailed distributions which includes the Weibull distributions with parameter $\alpha < 1/2$. The tails of the distributions considered in [4] and [19] are heavier than $e^{-\sqrt{t}}$. As is shown in [3] (see also [18]), the latter condition is crucial for the asymptotics (4) to hold.

The method proposed in this paper allows to find asymptotics for $\mathsf{P}\{bp(x) > t\}$ in the $M/G/1$ queue for both light- and heavy-tailed distributions. Moreover, we are able to obtain these asymptotics when $\mathsf{P}\{B > t\}$ is lighter than $e^{-\sqrt{t}}$ but still heavier than any exponential distribution. Using the results on tail asymptotics of the distribution of $bp(x)$, we can also obtain the results for the tail asymptotics of the distribution of bp.

The paper is organised as follows. In section 1 we present Theorems 1.1 and 1.2 that reduce the problem of finding asymptotics for $\mathsf{P}\{\tau_x > t\}$ and $\mathsf{P}\{\nu_x > n\}$ to studying asymptotics of $\mathsf{P}\{S_n \geq 0\}$. In section 2 we consider 4 classes of distributions: heavy-tailed distributions I (with tails heavier than $e^{-\sqrt{t}}$), heavy-tailed distributions II (with tails lighter than $e^{-\sqrt{t}}$), distributions satisfying Cramér's condition and distributions forming an intermediate case (distributions with light tails not satisfying Cramér's condition). For each of these cases we give known results on asymptotics of $\mathsf{P}\{S_n \geq 0\}$, show that the conditions of Theorem 1.2 are satisfied and hence, obtain results on the tail asymptotics of the distributions of τ_x and ν_x. Appendix A is devoted to the proofs of Theorems 1.1 and 1.2, in Appendix B we present some known results on Lévy processes that are used in our paper.

1   Main results

In this section we present Theorems 1.1 and 1.2 which connect the asymptotics for τ_x and ν_x with the asymptotics for $\mathsf{P}\{S_n \geq 0\}$. Before stating general Theorems, we need to introduce

Definition 1.1. A function $f : \mathbf{R}^+ \to \mathbf{R}^+$ belongs to the class $\mathcal{S}d(\gamma)$ with $\gamma \geq 0$ if, starting from some moment t, $f(t) > 0$ and

$$\lim_{t\to\infty} f(t-y)/f(t) = e^{\gamma y}, \quad y \in \mathbf{R}; \tag{5}$$

$$\lim_{t\to\infty} \frac{f^{*2}(t)}{f(t)} = \lim_{t\to\infty} \frac{\int_0^t f(t-y)f(y)\mathrm{d}y}{f(t)} = 2d = 2\int_0^\infty e^{\gamma y} f(y)\mathrm{d}y. \qquad (6)$$

The class $\mathcal{S}d := \mathcal{S}d(0)$ is called the class of subexponential densities.

A discrete-time analogue of this definition is

Definition 1.2. A sequence $\{a_n\}_{n\geq 0}$ belongs to the class $\mathcal{S}s(\gamma)$ with $\gamma \geq 0$ if starting from some index n, $a_n > 0$ and

$$\lim_{n\to\infty} a_{n-1}/a_n = e^\gamma, \qquad (7)$$

$$\lim_{n\to\infty} \frac{a_n^{*2}}{a_n} \equiv \lim_{n\to\infty} \frac{\sum_{i=0}^n a_i a_{n-i}}{a_n} = 2d = 2\sum_{i=0}^\infty a_i e^{\gamma i}. \qquad (8)$$

The class $\mathcal{S}s := \mathcal{S}s(0)$ is called the class of subexponential sequences.

Theorem 1.1. *Let the function*

$$\frac{\mathsf{P}\{X_t \geq 0\}}{t}, \quad t \geq 1$$

belong to the class $\mathcal{S}d(\gamma)$. In addition assume that for some $\alpha \geq 0$

$$\lim_{t\to\infty} \frac{\mathsf{P}\{X_t \geq 0\}}{\mathsf{P}\{X_t \geq y\}} = e^{\alpha y}, \quad \text{for any fixed } y. \qquad (9)$$

Then,

$$\mathsf{P}\{\tau_x > t\} \sim V(x)\frac{\mathsf{P}\{X_t \geq 0\}}{t}, \qquad (10)$$

for any x, a point of continuity of the function

$$V(x) \equiv \begin{cases} \mathsf{E}\{\tau_x\}, & \gamma = \alpha = 0 \\ e^{\alpha x}\int_0^\infty e^{\gamma t}\mathsf{E}\{e^{\alpha N_t}; |N_t| \leq x\}\mathrm{d}t, & \text{otherwise} \end{cases} \qquad (11)$$

where $N_t = \inf_{0\leq s\leq t} X_s$.

In the next Theorem we show that it is possible to obtain asymptotics for τ_x and ν_x under the same conditions for both random walks and Lévy processes.

Theorem 1.2. *Let X_t be either a Lévy process or a random walk. Let the sequence*

$$\frac{\mathsf{P}\{X_n \geq 0\}}{n}, \quad n \in \mathbf{N}$$

belong to the class $\mathcal{S}s(\gamma)$. In addition assume that for some $\alpha \geq 0$

$$\lim_{n\to\infty} \frac{\mathsf{P}\{X_n \geq 0\}}{\mathsf{P}\{X_n \geq y\}} = e^{\alpha y}, \quad \text{for any fixed } y, \quad n \in \mathbf{N}. \qquad (12)$$

Then, if X_t is a Lévy process, for any x,

$$\mathsf{P}\{\tau_x > t\} \quad \sim \quad V(x)e^{-\gamma(t-[t])}\frac{\mathsf{P}\{X_{[t]} \geq 0\}}{t},$$

where $V(x)$ is defined in Theorem 1.1.

If X_n is a random walk, then

$$\mathsf{P}\{\tau_x > n\} \quad \sim \quad V_{rw}(x)\frac{\mathsf{P}\{X_n \geq 0\}}{n},$$

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where

$$V_{rw}(x) = \begin{cases} \mathbf{E}\nu_x, & \gamma = \alpha = 0 \\ e^{\alpha x} \sum_{k=0}^{\infty} e^{\gamma k} \mathbf{E}\{e^{\alpha N_k}; |N_k| \leq x\}, & otherwise \end{cases}$$

where $N_k = \min\limits_{0 \leq l \leq k} X_l$.

Remark 1. The conditions of Theorem 1.1 and (or) conditions of Theorem 1.2 imply that $e^{-\gamma} = \mathsf{E}\{e^{\alpha X_1}\}$. The proof of this fact is given in Appendix A. Note also that this fact implies that $\alpha = 0$ if and only if $\gamma = 0$. This corresponds to the subexponential case.

2 Explicit results

This section consists of 4 subsections. Each of these subsections is devoted to a class of distributions for which we present known results on large deviations of sums of random variables, with the help of these results we show that the conditions of Theorem 1.2 are satisfied and as a result we obtain the asymptotics for the tail distributions of τ_x and ν_x. First, we prove Theorem 2.2 in which we study the case when $-\ln \mathsf{P}\{X_1 > t\} = o(\sqrt{t})$. Further, in Theorem 2.4, we analyse the case when $-\ln \mathsf{P}\{X_1 > t\}$ is regularly varying with parameter $\alpha \in [1/2, 1)$. For the first two cases, we use some results from paper [12]. Then, in Theorem 2.5 we give the asymptotics for Cramér's case. This includes (partially) the distributions with exponential tails and tails which are lighter than exponential. Finally, in Theorem 2.7, we analyse distributions with exponential tails that are not covered by Cramér's case. As corollaries we give corresponding results for the tail asymptotics of the busy period of an $M/G/1$ queue.

2.1 Heavy-tailed distributions I

The following Theorem easily follows from [12, Corollary 2.1].

Theorem 2.1. *Let $S_n = \sum\limits_{i=1}^{n} \xi_i$ be a random walk. Let $\mathsf{E}\{\xi_1\} = 0$ and $\mathsf{E}\{|\xi_1|^\kappa\} < \infty$ for some $\kappa \in (1, 2]$. Assume that*

$$\frac{\overline{F}(n - n^{1/\kappa})}{\overline{F}(n)} \to 1 \tag{13}$$

as $n \to \infty$ and assume also that $x \to x^\kappa \mathbf{P}(\xi_1 > x)$ either belongs to $\mathcal{S}d(0)$ or is O-regularly varying. Then for any $a > 0$,

$$\mathsf{P}\{S_n > na\} \sim n\mathsf{P}\{\xi_1 > na\} \tag{14}$$

as $n \to \infty$.

For the tail asymptotics of τ_x and ν_x the following is true.

Theorem 2.2. *Let X_t be either a Lévy process or a random walk. Assume that the distribution of X_1 satisfies the conditions of Theorem 2.1. Let $\mathsf{E}\{X_1\} = -a < 0$. Then*

$$\mathsf{P}\{\tau_x > t\} \quad \sim \quad \mathsf{E}\{\tau_x\}\mathsf{P}\{X_1 > ta\} \sim \mathsf{E}\{\tau_x\}\overline{\Pi}(ta), \quad t \to \infty; \tag{15}$$

$$\mathsf{P}\{\nu_x > n\} \quad \sim \quad \mathsf{E}\{\nu_x\}\mathsf{P}\{X_1 > na\}, \quad n \to \infty. \tag{16}$$

Remark 2. Note that the conditions of the latter Theorem imply that $\overline{F}(y - \sqrt{y}) \sim \overline{F}(y)$. It, in turn, implies that $-\ln \overline{F}(y) = o(\sqrt{y})$. Thus, we again have the Weibull distribution with parameter $1/2$ as a boundary.

Remark 3. The proof of Theorem 2.2 relies on Theorem 2.1 which is a result of [12]. However, the same asymptotics for τ_x and ν_x may be obtained for any distributions satisfying the asymptotic equivalence $\mathsf{P}\{S_n > na\} \sim n\mathsf{P}\{\xi_1 > na\}$. For instance, from the results of [25] it follows that such asymptotics hold for regularly varying distributions. The results of [32] imply that the same holds for Weibull-type distributions with a parameter smaller than $1/2$. In [12] it is shown that Theorem 2.1 includes all the results known beforehand.

PROOF OF THEOREM 2.2. We should check the conditions of Theorem 1.2. First, it follows from Theorem 2.1 that $\mathsf{P}\{X_n \geq 0\} \sim \mathsf{P}\{X_{n+1} \geq 0\}$ and $\mathsf{P}\{X_n \geq y\} \sim \mathsf{P}\{X_n \geq 0\}$. It is then easy to check that $\mathsf{P}\{X_n \geq 0\}/n$ is a subexponential sequence.

Then, it follows from Theorem 2.1 and Theorem 1.2 that

$$\mathsf{P}\{X_t \geq 0\} \sim \mathsf{P}\{X_{[t]} \geq 0\} = \mathsf{P}\{X_{[t]} + [t]a \geq [t]a\} \sim [t]\mathsf{P}\{X_1 \geq [t]a\}.$$

Then,

$$\mathsf{P}\{\tau_x > t\} \sim \mathsf{E}\{\tau_x\}\frac{\mathsf{P}\{X_t \geq 0\}}{t} \sim \mathsf{E}\{\tau_x\}\mathsf{P}\{X_1 \geq [t]a\} \sim \mathsf{E}\{\tau_x\}\mathsf{P}\{X_1 > ta\}.$$

The proof of Theorem 2.2 is complete.

We also present a direct corollary of Theorem 2.2.

Corollary 2.1. *Let $\mathsf{E}\{B_1^\kappa\} < \infty$ and $\mathsf{E}\{A_1^\kappa\} < \infty$ for some $\kappa \in (1,2]$. Assume that the distribution of B_1 satisfies the conditions of Theorem 2.1. Then*

$$\mathsf{P}\{bp(x) > t\} \sim \frac{x}{\mathbf{E}A_1}\frac{1}{1-\rho}\mathsf{P}\{B_1 > (1-\rho)t\} \tag{17}$$

as $t \to \infty$ for any fixed $x > 0$.

PROOF OF COROLLARY 2.1. In the case of an $M/G/1$ system, case $X_1 = \sum_{i=1}^{N(1)} X_i - 1$ (see (3)). Hence, $\mathsf{E}\{X_1\} = \rho - 1$ and $\mathsf{P}(X_1 > t) \sim \frac{1}{\mathbf{E}A_1}\mathsf{P}(B_1 > t)$ as $t \to \infty$. Theorem 2.2 yields that

$$\mathsf{P}\{bp(x) > t\} \sim \mathsf{E}\{bp(x)\}\mathsf{P}\{B_1 > (1-\rho)t\}$$

as $t \to \infty$. It remains to note that $\mathbf{E}bp(x) = \frac{x}{1-\rho}$ (this, for example, can be obtained from [11, p.261, (4.94)]). The proof of Corollary 2.1 is complete.

2.2 Heavy-tailed distributions II

Denote $g(x) = -\ln \overline{F}(x)$. In this subsection we consider the case

$$\limsup \frac{g(x)}{\sqrt{x}} > 0.$$

For this we use [32, theorem 5a].

Theorem 2.3. *([32, theorem 5a]) Let $\mathsf{P}\{\xi_1 > y\} \sim e^{-g(y)}$ as $y \to \infty$ with a doubly differentiable function g such that $g''(y)$ does not decrease for $y \geq y_o$ and $yg''(y) \sim (\beta-1)g'(y)$ as $y \to \infty$ for some $\beta \in (0,1)$. Let*

$$k = \max\left\{l \in \{1,2,\ldots\} : \limsup_{z\to\infty} \frac{g(z)}{z^{l/(l+1)}} > 0\right\}. \tag{18}$$

Let

$$\mathsf{E}\{\xi\} = 0, \mathsf{E}\{\xi^2\} = 1, \mathsf{E}\{|\xi|^{k+3}\} < \infty,$$

$$R(y) = g(y) + \frac{(t-y)^2}{2n} - \sum_{i=1}^{k} \lambda_{i-1} \frac{(t-y)^{i+2}}{n^{i+1}}. \tag{19}$$

Let y_* be the maximal solution of $R'(y) = 0$. Then $y^* \leq t - \sqrt{n}$ and

$$\mathsf{P}\{S_n > t\} \sim n\sqrt{\frac{1}{nR''(y_*)}} \exp\{-R(y_*)\}, \quad n \to \infty \tag{20}$$

uniformly in $t > 1.6\eta(n)$, where $\eta(z)$ is such that $\eta^2(z)/g(\eta(z)) \sim z, z \to \infty$. Here, λ_i are the coefficients of the Cramér series (see [30] for the definition).

Remark 4. Note that the conditions of Theorem 2.3 imply that $g''(y)$ is a regularly varying function with parameter $(\beta - 2)$. This fact follows from the monotonicity of g'' and Karamata's Theorem. Then, $g'(y)$ is regularly varying with parameter $(\beta - 1)$ and $g(y)$ is regularly varying with parameter β. Also, under these conditions $\eta(z)$ may be equivalently defined as a function such that $|g''(\eta(z))| \sim \beta(1 - \beta)/z, z \to \infty$. Therefore, $\eta(z)$ is a monotone regularly varying function with parameter $1/(2 - \beta)$.

Remark 5. In the statement of [32, theorem 5a] it is not said that $y^* \leq t - \sqrt{n}$, but one can find this assertion in the proof of [32, Lemma 3a].

We find it difficult to apply Theorem 2.3 directly, since it gives the asymptotics in terms of the maximal solution to an equation. Therefore, we use the approach developed in [18] to simplify this equation.

Lemma 2.1. *Let all conditions of Theorem 2.3 hold. Let $t_n \to \infty$ be a sequence such that $t_n \geq 1.6\eta(n)$. Let y_n be any sequence such that $y_n \sim t_n$ and*

$$R'(y_n) = o(1/\sqrt{n}). \tag{21}$$

Then,

$$\mathsf{P}\{S_n > t_n\} \sim n\sqrt{\frac{1}{nR''(y_n)}} \exp\{-R(y_n)\}, \quad n \to \infty.$$

Also, for any sequence $\varepsilon_n = o(\sqrt{n})$,

$$\mathsf{P}\{S_n > t_n\} \sim \mathsf{P}\{S_n > t_n + \varepsilon_n\}.$$

PROOF OF LEMMA 2.1. First, we note that since g'' is monotone and regularly varying it is true that

$$R'(y + z) - R'(y) = zR''(y)(1 + o(1)), \quad y \to \infty, z = o(y).$$

Also, $R''(y_n) = (g''(y_n) + 1/n)(1 + o(1))$ and

$$|g''(y_n)| \leq |g''(1 + o(1)t_n)| \leq (1 + o(1))|g''(1.6\eta(n))|$$
$$\leq (1 + o(1))|g''(\eta(n))| = (1 + o(1))\beta(1 - \beta)/n \leq 1/(4n)$$

Then, for any $\varepsilon > 0$,

$$R'(y_n + \varepsilon\sqrt{n}) = R'(y_n) + \varepsilon\sqrt{n}R''(y_n) \geq o(1/\sqrt{n}) + 3/4\varepsilon/\sqrt{n} > 0$$
$$R'(y_n - \varepsilon\sqrt{n}) = R'(y_n) - \varepsilon\sqrt{n}R''(y_n) \leq o(1/\sqrt{n}) - 1/4\varepsilon/\sqrt{n} < 0.$$

Since R' is continuous there exists a sequence $\beta_n \in (y_n - o(\sqrt{n}), y_n + o(\sqrt{n}))$ such that $R'(\beta_n) = 0$ and $\beta_n \sim t_n$. Further, if there exists some other solution $\beta'_n > \beta_n$, then with necessity $\beta'_n \sim t_n \sim \beta_n$. But this is not possible since $R''(y)$ is positive on the interval (β_n, t_n).

To prove the first statement of the lemma, note

$$R(y_n) - R(\beta_n) = R'(\beta_n)(\beta_n - y_n) + (1 + o(1))R''(\beta_n)\frac{(\beta_n - y_n)^2}{2}$$
$$= (1 + o(1))R''(\beta_n)\frac{(\beta_n - y_n)^2}{2} \sim O(\frac{1}{n})o(\sqrt{n})^2 = o(1).$$

To prove the second statement of the lemma we should note that if β_n is a solution sequence for the equation $R_1(\beta_n) = 0$ for the first sequence t_n, then the corresponding equation for the sequence $t_n + \varepsilon_n$ is $R_2(\beta_n) = o(1/\sqrt{n})$. Then we should just apply the first statement of the lemma. The proof of lemma 2.1 is complete.

We shall now concentrate on the case $t_n = na$, the case needed for our purposes.

Corollary 2.2. *Let all conditions of Theorem 2.3 hold. Let $t_n = na$, where $a > 0$. Let y_n be any sequence such that $y_n \sim t_n$ and Condition (21) holds. Then,*

$$\mathsf{P}\{S_n > na\} \sim n \exp\{-R(y_n)\}, \quad n \to \infty.$$

Also, for any sequence $\varepsilon_n = o(\sqrt{n})$,

$$\mathsf{P}\{S_n > na\} \sim \mathsf{P}\{S_n > na + \varepsilon_n\}.$$

PROOF. This is just a reformulation of Lemma 2.1. We should just note that in this case $R''(y_n) \sim 1/n$.

Lemma 2.2. *Under the conditions of Theorem 2.3 let $t_n = na$. Define a sequence*

$$y_n^{(0)} = na, \quad y_n^{(j)} = y_n^{(j-1)} - nR'(y_n^{(j-1)}).$$

Then, for any $j \geq 1/(2k)$,

$$\mathsf{P}\{S_n > na\} = n \exp\{-R(y_n^{(j)})\}.$$

PROOF OF LEMMA 2.2. We have

$$|y_n^{(1)} - y_n^{(0)}| = n|R'(na)| = ng'(na).$$

This implies that $y_n^{(2)} \sim na$. Assume that we proved $y_n^{(i)} \sim na$ for all $i < j$. Then, using regular variation of g'' we obtain,

$$|y_n^{(j)} - y_n^{(j-1)}| = n|R'(y_n^{j-1}) - R'(y_n^{j-2})|$$
$$= (1 + o(1))n|g'(y_n^{j-1}) - g'(y_n^{j-2})| = (1 + o(1))n|g''(na)||y_n^{j-1} - y_n^{j-2}| = o(n). \quad (22)$$

Therefore, we can argue by induction that for $j \geq 1$,

$$|y_n^{(j)} - y_n^{(j-1)}| = O(1)(n|g''(na)|)^{j-1}ng'(na) = O(1)n(g'(n))^j = O(1)n\left(\frac{g(n)}{n}\right)^j.$$

Now make use of Condition (18), then

$$|y_n^{(j+1)} - y_n^{(j)}| = o(1)n\left(\frac{n^{1-1/(k+2)}}{n}\right)^{j+1} = o(1)n^{1-(j+1)/(k+2)} = o(\sqrt{n}), \quad (23)$$

provided $j \geq k/2$. Then

$$R'(y_n^{(j)}) = \frac{y_n^{(j)} - y_n^{(j+1)}}{n} = \frac{o(\sqrt{n})}{n} = o(1/\sqrt{n}).$$

The proof of Lemma 2.2 is complete.

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Lemma 2.3. *Assume that all conditions of the previous lemma hold. Then the sequence $a_n = \dfrac{\mathsf{P}\{S_n \geq 0\}}{n}$ is subexponential.*

PROOF OF LEMMA 2.3.

It follows from lemma 2.2 that $a_n \sim e^{-R(y_n^{(j)})}$ for sufficiently large j. We shall prove that for any j,

$$\lim_{n \to \infty} \frac{nR'(y_n^{(j)})}{R(y_n^{(j)})} = \beta. \tag{24}$$

This will imply that the sequence a_n is subexponential due to the sufficient conditions given in [20]. We prove (24) by induction. If $j = 0$, then $R((y_n^{(j)}) = g(y_n^{(j)})$ and (24) holds since g is a regularly varying function with parameter β. Assume (24) holds for some j. Note that (23) implies that $R'(y_n^{(j)}) = \dfrac{y_n^{(j)} - y_n^{(j+1)}}{n}$ is regularly varying, and hence, taking (24) into account, $R(y_n^{(j)})$ is also regularly varying. Recall that $y_n^{(j)} \sim na$ for each j and that $y_n^{(j)} - y_n^{(j+1)} = o(n)$ (see (22)). Then $R(y_n^{(j+1)}) = R(y_n^{(j)} + (y_n^{(j+1)} - y_n^{(j)})) \sim R(y_n^{(j)})$ and also $R'(y_n^{(j+1)}) \sim R'(y_n^{(j)})$. Hence, (24) holds for $j + 1$ as well. The proof of Lemma 2.3 is complete.

We now give the result on the asymptotic behaviour of the tails of τ_x and ν_x.

Theorem 2.4. *Assume that $\{X_t\}$ is a Lévy process or a random walk such that $\mathsf{E}\{X_1\} = -a < 0$. Let all the conditions of Theorem 2.3 hold for the distribution of X_1. Let $R'(y(t)) = o(1/\sqrt{t})$. Then*

$$\mathsf{P}\{\tau_x > t\} \quad \sim \quad \mathsf{E}\{\tau_x\} \exp\{-R(y(t))\}, \quad t \to \infty, \tag{25}$$

$$\mathsf{P}\{\nu_x > n\} \quad \sim \quad \mathsf{E}\{\nu_x\} \exp\{-R(y(n))\}, \quad n \to \infty. \tag{26}$$

PROOF OF THEOREM 2.4. We use Theorem 1.2 again. First, it follows from Corollary 2.2 that $\mathsf{P}\{X_n > y\} \sim \mathsf{P}\{X_n > 0\} \sim \mathsf{P}\{X_{n+1} > 0\}$. Second, according to lemma 2.3, sequence $\alpha_n = \mathsf{P}\{X_n > 0\}/n$ is subexponential. Then, we can just apply Theorem 1.2. The proof of Theorem 2.4 is complete.

Corollary 2.3. *Let $\mathsf{P}\{B_1 > y\} \sim e^{-g(y)}$. Assume that g and B_1 satisfy all conditions of Theorem 2.4 with $a = 1 - \rho$. Then the asymptotics are given by*

$$\mathsf{P}\{bp(x) > t\} \quad \sim \quad \frac{x}{\mathsf{E}\{A\}(1 - \rho)} \exp\{-R(y(t))\}, \quad t \to \infty. \tag{27}$$

PROOF OF COROLLARY 2.3 repeats the proof of corollary 2.1.

One can apply Lemma 2.2 to obtain tail asymptotics for the distributions of τ_x and ν_x explicitly. In particular, the following corollary holds.

Corollary 2.4. *Under the conditions of Theorem 2.4 let $g(y) = o(y^{3/4})$. Then*

$$\mathsf{P}\{\tau_x > t\} \sim \mathsf{E}\{\tau_x\} \exp\{-R(ta - tg'(ta))\}, \quad t \to \infty.$$

If $\mathsf{P}\{B_1 > t\} \sim e^{-t^\beta}, 1/2 < \beta < 1$, then for $k = [1/(1 - \beta)]$ some positive constants $D_1, \ldots, D_k > 0$,

$$\mathsf{P}\{\tau_x > t\} \sim \mathsf{E}\{\tau_x\} \exp\{-(at)^\beta + D_1 t^{2\beta - 1} + \ldots + D_k t^{k\beta - k + 1}\}.$$

Similar corollaries can be formulated for ν_x and $bp(x)$.

2.3 Cramér's case

Let $m(s) = \mathsf{E}\{e^{sX_1}\}$ be the moment generating function of X_1.

Theorem 2.5. *Assume that $\{X_t\}$ is a Lévy process or a random walk such that $\mathsf{E}\{X_1\} = -a < 0$. Let solution α to the equation $m'(s) = 0$ exist and $m''(\alpha) < \infty$. Put $\gamma = \ln m(\alpha)$. Assume also that the distribution of X_1 is non-lattice. Then*

$$\mathsf{P}\{\tau_x > t\} \quad \sim \quad V(x)\frac{1}{\sqrt{2\pi}t^{3/2}\widehat{\sigma}(\alpha)\alpha}e^{-\gamma t}, \quad t \to \infty,$$

$$\mathsf{P}\{\nu_x > n\} \quad \sim \quad \frac{1}{\sqrt{2\pi}n^{3/2}\widehat{\sigma}(\alpha)\alpha}e^{-\gamma n}, \quad n \to \infty.$$

PROOF OF THEOREM 2.5. It follows from the Petrov Theorem (see [29, theorem 2]) that $a_n = \mathsf{P}\{X_n \geq 0\}/n \sim \frac{1}{\sqrt{2\pi}n^{3/2}\widehat{\sigma}(\alpha)\alpha}e^{-\gamma n}$. Then $a_n \in \mathcal{S}s(\gamma)$. Indeed, $a_{n-1}/a_n \to e^{\gamma}$ and for some constant C,

$$\sum_{k=1}^{n-1}\frac{a_k a_{n-k}}{a_n} \leq C\sum_{k=1}^{n-1}(\frac{n}{k(n-k)})^{3/2} \leq 4C\sum_{k=1}^{n-1}\frac{1}{k^{3/2}} < 4C\sum_{k=1}^{\infty}\frac{1}{k^{3/2}}.$$

It follows from the dominated convergence Theorem that $a_n^{*2}/a_n \to 2\sum_{n=1}^{\infty}e^{\gamma n}a_n$. Also, Petrov's Theorem implies that

$$\mathsf{P}\{X_n \geq y\}/\mathsf{P}\{X_n \geq 0\} \sim e^{-\alpha y}, \quad n \to \infty.$$

Therefore, we can apply Theorem 1.2 to obtain the needed asymptotics. The proof of Theorem 2.5 is complete.

Corollary 2.5. *Let $\alpha > 0$ be a solution to the equation $\lambda m'_B(\alpha) = 1$ such that $\widehat{\sigma}^2 = \lambda m''_B(\alpha) < \infty$. Put $\gamma = \alpha - \lambda(m_B(\alpha) - 1)$. Then,*

$$\mathsf{P}\{bp(x) > t\} \sim \frac{1}{\sqrt{2\pi\widehat{\sigma}^2}\gamma t^{3/2}}xe^{\alpha x}e^{-\gamma t} \tag{28}$$

PROOF OF COROLLARY 2.5. It is clear that α and γ are exactly the same as in Theorem 2.5. Therefore, we should just find $V(x)$. Since $e^{-\gamma} = \mathsf{E}\{e^{\alpha X_1}\}$, the process $\exp\{\alpha X_t + \gamma t\}$ is a martingale with mean 1. Then, since $X_{\tau_x} = -x$, we have

$$1 = \mathsf{E}\{e^{\alpha X_{\tau_x} + \gamma \tau_x}\} = e^{-\alpha x}\mathsf{E}\{e^{\gamma \tau_x}\}.$$

Then, $C(x) = \gamma^{-1}(\mathsf{E}\{e^{\gamma \tau_x}\} - 1) = \gamma^{-1}(e^{\alpha x} - 1)$, and it follows from (36) that

$$V(x) = C(x) + \alpha e^{\alpha x}\int_0^x e^{-\alpha y}C(y)\mathrm{d}y = \frac{\alpha}{\gamma}xe^{\alpha x}.$$

The proof of Corollary 2.5 is complete.

2.4 Intermediate case

We now proceed to the intermediate case, that is when equation the $m'(s) = 0$ does not have a positive solution but $m(s) < \infty$ for some $s > 0$. In this case we shall assume that $\mathsf{P}\{\xi_1 > t\} = e^{-\alpha t}\overline{G}(t)$ for all positive t where $\alpha > 0$ and $\overline{G}(t)$ is a tail of some heavy-tailed distribution. Introduce the random walk $\{\tilde{S}_n\}$ (called the adjunct random walk in [6]) whose increments have the distribution

$$\tilde{F}(dy) = \frac{1}{m(\alpha)}e^{\alpha y}F(dy)$$

and denote $\delta = -\mathsf{E}\{\tilde{S}_1\}$. The following result on large deviations may be found in [12]. It is a generalisation of the result of [6] where asymptotics for the large deviations probabilities are obtained under the assumption that $\overline{G}(t)$ is a regularly varying function.

Theorem 2.6. *Assume that* $\mathsf{P}\{\xi_1 > t\} = e^{-\alpha t}\overline{G}(t)$ *for all positive* t *where* $\alpha > 0$ *and* $\overline{G}(t)$ *satisfies conditions of Theorem 2.1. Let* $\mathsf{E}\{\xi_1\} < 0$, *and* $m'(s) \neq 0$ *for* $0 < s \leq \alpha$. *Assume also that* $\delta < \infty$. *Put* $e^{-\gamma} = m(\alpha)$. *Then uniformly in* x *such that* $x \geq -n(\delta - \varepsilon)$

$$\mathsf{P}\{S_n > x\} \sim \frac{1}{m(\alpha)}e^{-\gamma n}e^{-\alpha x}n\overline{G}(x + n\delta).$$

Remark 6. It is easy to see that the conditions of Theorem 2.6 imply that $\delta > 0$.

Using the latter Theorem, one can obtain the following result for the tail asymptotics of τ_x and ν_x.

Theorem 2.7. *Assume that* $\{X_t\}$ *is a Lévy process or a random walk such that the distribution of* X_1 *satisfies the conditions of Theorem 2.6. Then*

$$\mathsf{P}\{\tau_x > t\} \sim V(x)\frac{1}{m(\alpha)}e^{-\gamma t}\overline{G}(t\delta)$$

and

$$\mathsf{P}\{\nu_x > n\} \sim V_{rw}(x)\frac{1}{m(\alpha)}e^{-\gamma n}\overline{G}(n\delta).$$

PROOF OF THEOREM 2.7.

Theorem 2.6 implies that

$$\mathsf{P}\{X_n \geq 0\} \sim \frac{1}{m(\alpha)}e^{-\gamma n}n\overline{G}(n\delta)$$

and

$$\mathsf{P}\{X_n \geq y\} \sim \frac{1}{m(\alpha)}e^{-\gamma n}n\overline{G}(n\delta + y)e^{-y\alpha}.$$

The conditions of Theorem 1.2 can now be checked in a straightforward manner. The proof of Theorem 2.7 is complete.

Remark 7. To the best of our knowledge, the only result on large deviations of sums of random variables belonging to the intermediate case is contained in [6]. The result presented in this paper concerns distributions F such that the function $e^{\alpha x}\overline{F}(x)$ is regularly varying with parameter $-\beta, 2 < \beta < \infty$. The analogue of Theorem 2.7 for such distributions may be obtained using Lemma 3 from [6] instead of our Theorem 2.6.

A   Proofs of Theorems 1.1 and 1.2

PROOF OF REMARK 1. Indeed, prove first that $Ee^{\alpha X_1} \leq e^{-\gamma}$. Fix arbitrary $C > 0$. Then for large enough t, uniformly in $y \in (-C, C)$,

$$e^{\alpha y} \leq (1 + \varepsilon)\frac{P(X_t > -y)}{P(X_t > 0)}.$$

Consider now

$$\int_{-C}^{C} e^{\alpha y}P(X_1 \in dy) \leq (1 + \varepsilon)\frac{\int_{-C}^{C} P(X_t > -y)P(X_1 \in dy)}{P(X_t > 0)}$$

$$= (1 + \varepsilon)\frac{\int_{-C}^{C} P(X_t > -y)P(X_{t+1} - X_t \in dy)}{P(X_t > 0)}$$

$$= (1 + \varepsilon)\frac{P(X_{t+1} - X_t \in (-C, C), X_{t+1} > 0)}{P(X_t > 0)} \leq \frac{P(X_{t+1} > 0)}{P(X_t > 0)} \leq (1 + \varepsilon)^2 e^{-\gamma}$$

for large enough t. Since ε is an arbitrary positive number, we have $\int_{-C}^{C} e^{\alpha y} P(X_1 \in dy) \leq e^{-\gamma}$ and hence, $Ee^{\alpha X_1} \leq e^{-\gamma}$.

The inequality $Ee^{\alpha X_1} \geq e^{-\gamma}$ can be proved in a similar way. Take arbitrary $C > 0$ and consider

$$\int_{-C}^{C} \frac{P(X_t > -y)P(X_{t+1} - X_t \in dy)}{P(X_t > 0)} \leq (1+\varepsilon) \int_{-C}^{C} e^{\alpha y} P(X_1 \in dy) \leq (1+\varepsilon)Ee^{\alpha X_1}$$

for sufficiently large t. Since C is arbitrary, we have

$$\frac{P(X_{t+1} > 0)}{P(X_t > 0)} = \int_{-\infty}^{\infty} \frac{P(X_t > -y)P(X_{t+1} - X_t \in dy)}{P(X_t > 0)} \leq (1+\varepsilon)Ee^{\alpha X_1}.$$

We also have that for sufficiently large t

$$e^{-\gamma} \leq (1+\varepsilon)\frac{P(X_{t+1} > 0)}{P(X_t > 0)} \leq (1+\varepsilon)^2 Ee^{\alpha X_1}$$

which concludes the proof since the LHS does not depend on ε. The proof of Remark 1 is complete.

PROOF OF THEOREM 1.1. Recall that $N_t = \inf_{s \leq t} X_s$. It is clear that $\mathsf{P}\{\tau_x > t\} = \mathsf{P}\{|N_t| \leq x\}$. Our starting point is the formula which follows from the Wiener-Hopf identity for Lévy processes, see [33, (47.9)] (with obvious changes: we should substitute the infimum process instead of the supremum process). Thus, for $q > 0$ and $u \geq 0$,

$$q \int_0^{\infty} e^{-qt} \mathsf{E}\{e^{uN_t}\}dt = \exp\left\{\int_0^{\infty} t^{-1} e^{-qt} dt \int_{(-\infty,0)} (e^{uy} - 1)\mathsf{P}\{X_t \in dy\}\right\}. \tag{29}$$

For $q > 0$, make use of the Frullani integral

$$-\ln q = \int_0^{\infty} \frac{e^{-qt} - e^{-t}}{t} dt$$

and rewrite (29) in the following form

$$\int_0^{\infty} e^{-qt} \mathsf{E}\{e^{uN_t}\}dt = \exp\left\{\int_0^{\infty} \frac{e^{-qt} - e^{-t}}{t} dt \right.$$
$$\left. + \int_0^{\infty} t^{-1} e^{-qt} dt \int_{(-\infty,0)} (e^{uy} - 1)\mathsf{P}\{X_t \in dy\}\right\}. \tag{30}$$

First we want to show that the right-hand side in (30) converges as $q \to 0$. For that, let us represent the exponent on the right-hand side of (30) as follows,

$$\int_0^1 \frac{e^{-qt} - 1}{t} dt + \int_0^1 \frac{1 - e^{-t}}{t} dt - \int_1^{\infty} \frac{e^{-t}}{t} dt$$
$$+ \int_1^{\infty} t^{-1} e^{-qt} \left\{1 + \int_{(-\infty,0)} (e^{uy} - 1)\mathbf{P}(X_t \in dy)\right\} dt$$
$$+ \int_0^1 t^{-1} e^{-qt} dt \int_{(-\infty,0)} (e^{uy} - 1)\mathbf{P}(X_t \in dy). \tag{31}$$

It is clear that the first integral converges to 0 as $q \to 0$. The second and third integral are well defined and constant. The fourth and fifth integral are monotone in q, and therefore it is sufficient to prove that they are

finite for $q = 0$ and then to apply the monotone convergence Theorem. For $q = 0$, the fourth integral is equal to

$$\int_1^\infty t^{-1} \left\{ 1 + \int_{(-\infty,0)} (e^{uy} - 1)\mathsf{P}\{X_t \in dy\} \right\} dt$$

$$= \int_1^\infty t^{-1} \left\{ \mathsf{P}\{X_t \geq 0\} + \int_{(-\infty,0)} e^{uy}\mathsf{P}\{X_t \in dy\} \right\} dt. \quad (32)$$

To deal with the second term in (32), we prove

Lemma A.1. *Let function* $\mathsf{P}\{X_t \geq 0\}/t$ *be such that Condition (5) holds. Assume also that condition (9) of Theorem 1.1 holds. Then, there exists* u_0 *such that for any* $u > u_0$,

$$\int_{(-\infty,0)} e^{uy}\mathsf{P}\{X_t \in dy\} \sim \frac{\alpha}{u - \alpha}\mathsf{P}\{X_t \geq 0\}. \quad (33)$$

Remark. We use the convention that $a(x) \sim 0 \cdot b(x)$ means that $a(x) = o(b(x))$.

PROOF OF LEMMA A.1. Using integration by parts we obtain that

$$\int_{(-\infty,0)} e^{uy}\mathsf{P}\{X_t \in dy\} = u \int_0^\infty (\mathsf{P}\{X_t \geq -y\} - \mathsf{P}\{X_t \geq 0\})e^{-uy}dy.$$

We can pick function $h(t) \uparrow \infty$ such that (9) holds uniformly in $y \in [-h(t), 0]$. Then, for $u > \alpha$,

$$u \int_0^{h(t)} (\mathsf{P}\{X_t \geq -y\} - \mathsf{P}\{X_t \geq 0\})e^{-uy}dy$$

$$= (1 + o(1))\mathsf{P}\{X_t \geq 0\}u \int_0^{h(t)} (e^{\alpha y} - 1)e^{-uy}dy = \frac{\alpha + o(1)}{u - \alpha}\mathsf{P}\{X_t \geq 0\}$$

Further, since $\mathsf{P}\{X_t \geq 0\}/t$ satisfies Condition (5), for sufficiently large t,

$$\mathsf{P}\{X_{t+n} \geq 0\} \leq e^n \mathsf{P}\{X_t \geq 0\}, \quad n \geq 1. \quad (34)$$

Since $\mathsf{P}\{X_1 > 0\} > 0$, then for some $\delta = 1/l > 0$, $\mathsf{P}\{X_1 \geq \delta\} > 0$, where l is a positive integer. Then,

$$\mathsf{P}\{X_{ln} \geq n\} \geq \mathsf{P}\{X_1 \geq \delta, X_2 - X_1 \geq \delta, \dots X_{ln} - X_{ln-1} \geq \delta\} = \mathsf{P}\{X_1 \geq \delta\}^{ln}.$$

Let u_0 be such that $e^{-u_0} = e^{-2l}\mathsf{P}\{X_1 \geq \delta\}^l$. Then, for all $u > u_0$,

$$e^{-un} \leq e^{-u_0 n} = e^{-2ln}\mathsf{P}\{X_1 \geq \delta\}^{ln} \leq e^{-2ln}\mathsf{P}\{X_{ln} \geq n\}.$$

Therefore, for $u > u_0$,

$$\int_{-\infty}^{-h(t)} (\mathsf{P}\{X_t \geq -y\} - \mathsf{P}\{X_t \geq 0\})e^{-uy}dy \leq \sum_{n=[h(t)]}^\infty e^{-un}\mathsf{P}\{X_t \geq -n\}$$

$$\leq \sum_{n=[h(t)]}^\infty e^{-2ln}\mathsf{P}\{X_{ln} \geq n\}\mathsf{P}\{X_t \geq -n\}$$

$$= \sum_{n=[h(t)]}^\infty e^{-2ln}\mathsf{P}\{X_{t+ln} - X_t \geq n, X_t \geq -n\} \leq \sum_{n=[h(t)]}^\infty e^{-2ln}\mathsf{P}\{X_{t+ln} \geq 0\}.$$

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It follows from (34) that, as $t \to \infty$,

$$\sum_{n=[h(t)]}^{\infty} e^{-2ln} P\{X_{t+ln} \geq 0\} \leq P\{X_t \geq 0\} \sum_{n=[h(t)]}^{\infty} e^{-ln} = o(P\{X_t \geq 0\}).$$

The proof of Lemma A.1 is complete.

We may now continue to analyse (32). It follows from Lemma A.1 that for some constant $C > 0$,

$$\int_1^{\infty} t^{-1} \left\{ P\{X_t \geq 0\} + \int_{(-\infty,0)} e^{uy} P\{X_t \in dy\} \right\} dt \leq C \int_1^{\infty} t^{-1} P\{X_t \geq 0\} dt < \infty.$$

The finiteness of the latter integral follows from (2). We now proceed to the last term in (31). Making use of the inequality: $1 - e^{-x} \leq x$, for all $x \geq 0$, we obtain

$$\int_0^1 t^{-1} dt \int_{(-\infty,0)} (1 - e^{uy}) \mathbf{P}(X_t \in dy)$$

$$\leq \int_0^1 t^{-1} dt \left(\int_{(-1,0)} (-uy) \mathbf{P}(X_t \in dy) + P\{X_t < -1\} \right)$$

$$= \int_0^1 E\{u X_t \mathbf{I}(X_t \in [-1,0])\} t^{-1} dt + \int_0^1 t^{-1} dt P\{X_t < -1\} < \infty.$$

The finiteness of the latter integral follows from the estimates in Lemma B.1 in Appendix A. Therefore, the last term in (31) converges by the monotone convergence Theorem.

Now, letting $q \to 0$ in (30), we have,

$$\int_0^{\infty} E\{e^{uN_t}\} dt < \infty.$$

For fixed u, let now

$$f_u(t) = \frac{E\{e^{uN_t}\}}{\int_0^{\infty} E\{e^{uN_t}\} dt}$$

be the density of a random variable Z. Then, using representations (31) and (32), we may rewrite (30)

$$E\{e^{-qZ}\} = \exp\left\{ \int_0^1 \frac{e^{-qt} - 1}{t} dt + \int_1^{\infty} \frac{e^{-qt} - 1}{t} \left(P\{X_t \geq 0\} + \int_{(-\infty,0)} e^{uy} P\{X_t \in dy\} \right) dt \right.$$

$$\left. + \int_0^1 \frac{e^{-qt} - 1}{t} dt \int_{(-\infty,0)} (e^{uy} - 1) \mathbf{P}(X_t \in dy) \right\}$$

$$\equiv \exp\left\{ \int_0^1 (e^{-qt} - 1) \nu_1(dt) + \int_1^{\infty} (e^{-qt} - 1) \nu_2(dt) + \int_0^1 (e^{-qt} - 1) \nu_3(dt) \right\}.$$

Then Z is an infinitely divisible variable on $[0, \infty)$ with the Lévy measure $\nu(dt) = \nu_1(dt) + \nu_2(dt) + \nu_3(dt)$. Indeed, $\int_0^1 t \nu_1(dt) < \infty$. Further, as we have already shown $\nu_3(0,1) < \infty$. Finally, as follows from Lemma A.1,

$$f_2(t) \equiv \frac{d\nu_2}{dt} = \frac{P\{X_t \geq 0\}}{t} + \frac{1}{t} \int_{(-\infty,0)} e^{uy} P\{X_t \in dy\} \sim \frac{u}{u - \alpha} \frac{P\{X_t \geq 0\}}{t}, \quad t \to \infty.$$

Therefore, by Condition (2), $\nu_2(1, \infty) < \infty$. Now we are in the position to apply Theorem B.3 from the Appendix. Since the density of the Lévy measure $f(t) \sim P\{X_t \geq 0\}/t$ belongs to the class $\mathcal{S}d(\gamma)$, by Theorem B.3, we have for any fixed u,

$$f_u(t) \sim \left(\int_0^{\infty} e^{\gamma y} f_u(y) dy \right) \frac{u}{u - \alpha} \frac{P\{X_t \geq 0\}}{t}.$$

Equivalently, for all u,

$$\frac{\mathsf{E}\{e^{uN_t}\}}{\mathsf{P}\{X_t \geq 0\}/t} \to \frac{u}{u - \alpha} \int_0^\infty e^{\gamma t} \mathsf{E}\{e^{uN_t}\} \mathrm{dt}. \tag{35}$$

Then, changing the order of integration, we obtain

$$\int_0^\infty e^{\gamma t} \mathsf{E}\{e^{uN_t}\} \mathrm{dt} = \int_0^\infty e^{\gamma t} \left(\int_0^\infty e^{-ux} \mathsf{P}\{|N_t| \in \mathrm{dx}\} \right) \mathrm{dt} = \int_0^\infty e^{-ux} d_x C(x),$$

where $C(x) = \int_0^\infty e^{\gamma t} \mathsf{P}\{|N_t| \leq x\} \mathrm{dt}$. Therefore, (35) is equivalent to

$$\frac{\mathsf{E}\{e^{uN_t}\}}{\mathsf{P}\{X_t \geq 0\}/t} \to \frac{u}{u - \alpha} \int_0^\infty e^{-ux} d_x C(x).$$

Now note that $u/(u - \alpha)$ is the LST of the measure D which has unit mass at 0 and density $\alpha e^{\alpha y}$ on the positive half-line. Therefore,

$$\frac{\mathsf{P}\{|N_t| \leq x\}}{\mathsf{P}\{X_t \geq 0\}/t} \to D * C(x) \quad = \quad V(x) = C(x) + \alpha e^{\alpha x} \int_0^x e^{-\alpha y} C(y) \mathrm{dy} \tag{36}$$

$$= \quad e^{\alpha x} \int_0^\infty e^{\gamma t} \mathsf{E}\{e^{\alpha N_t}; |N_t| \leq x\} \mathrm{dt}$$

for all x, where function $V(x)$ is continuous. This is equivalent to

$$\frac{\mathsf{P}\{\tau_x > t\}}{\mathsf{P}\{X_t \geq 0\}/t} \to V(x).$$

Finally, it is clear that if $\alpha = \gamma = 0$ then

$$V(x) = \int_0^\infty \mathsf{P}\{|N_t| \leq x\} \mathrm{dt} = \int_0^\infty \mathsf{P}\{\tau_x > t\} \mathrm{dt} = \mathsf{E}\{\tau_x\}.$$

The proof of Theorem 1.1 is complete.

Now we will show that for Theorem 1.1 to hold it is sufficient that its conditions hold for positive integers t. This will allow us to reduce the problem of verifying properties of $X_t, t \in \mathbf{R}$ to verifying the corresponding properties of the random walk $X_n, n \in \mathbf{N}$.

Lemma A.2. *Assume that sequence* $\mathsf{P}\{X_n \geq 0\}/n, n \in \mathbf{N}$ *belongs to the class* $Ss(\gamma)$, *for any fixed* y

$$\mathsf{P}(X_n \geq 0) \sim e^{\alpha y} \mathsf{P}(X_n \geq y), \quad n \to \infty, \tag{37}$$

and $e^{-\gamma} = \mathsf{E}\{e^{\alpha X_1}\}$. *Then, function* $\mathsf{P}\{X_t \geq 0\}/t, t \in \mathbf{R}$ *belongs to the class* $\mathcal{S}d(\gamma)$ *and for any fixed* y,

$$\mathsf{P}(X_t \geq 0) \sim e^{\alpha y} \mathsf{P}(X_t \geq y), \quad t \in \mathbf{R}^+, t \to \infty. \tag{38}$$

Proof of Lemma A.2. First, we prove that for any $0 < \varepsilon < 1$,

$$\mathsf{P}\{X_{n+\varepsilon} \geq 0\} \sim e^{-\gamma\varepsilon} \mathsf{P}\{X_n \geq 0\}. \tag{39}$$

It is not difficult to prove that there exists a function $h(n) \uparrow \infty$ such that Condition (37) holds uniformly in $|z| \leq h(n)$. We start with the total probability formula

$$\mathsf{P}\{X_{n+\varepsilon} \geq 0\} \equiv P_1 + P_2 + P_3 = \mathsf{P}\{X_{n+\varepsilon} \geq 0, |X_{n+\varepsilon} - X_n| \leq h(n)\}$$
$$+ \mathsf{P}\{X_{n+\varepsilon} \geq 0, X_{n+\varepsilon} - X_n > h(n)\} + \mathsf{P}\{X_{n+\varepsilon} \geq 0, X_{n+\varepsilon} - X_n < -h(n)\}. \tag{40}$$

Then, since $\mathsf{E}\{e^{\alpha X_1}\} = e^{-\gamma}$,

$$P_1 = \int_{-h(n)}^{h(n)} \mathsf{P}\{X_\varepsilon \in \mathrm{d}y\}\mathsf{P}\{X_n \geq -y\} \sim \mathsf{P}\{X_n \geq 0\} \int_{-\infty}^{\infty} e^{\alpha y}\mathsf{P}\{X_\varepsilon \in \mathrm{d}y\} = e^{-\gamma\varepsilon}\mathsf{P}\{X_n \geq 0\}.$$

Before proceeding further, note that if we take $\varepsilon = 1$ then it follows from (7), (40) and the latter equivalence that

$$\mathsf{P}\{X_{n+1} \geq 0, X_{n+1} - X_n > h(n)\} = o(\mathsf{P}\{X_n \geq 0\}). \tag{41}$$

Further,

$$P_2 = \mathsf{P}\{X_n \geq -h(n)\}\mathsf{P}\{X_{n+\varepsilon} - X_n > h(n)\} + \int_{-\infty}^{-h(n)} \mathsf{P}\{X_n \in \mathrm{d}y\}\mathsf{P}\{X_{n+\varepsilon} - X_n \geq -y\}.$$

Now note that

$$\mathsf{P}\{X_1 \geq y\} \geq \mathsf{P}\{X_\varepsilon \geq y, X_1 - X_\varepsilon \geq 0\} = \mathsf{P}\{X_\varepsilon \geq y\}\mathsf{P}\{X_{1-\varepsilon} \geq 0\},$$

which implies that

$$P_2 \leq \frac{1}{\mathsf{P}\{X_{1-\varepsilon} \geq 0\}}(\mathsf{P}\{X_n \geq -h(n)\}\mathsf{P}\{X_{n+1} - X_n > h(n)\}$$
$$+ \int_{-\infty}^{-h(n)} \mathsf{P}\{X_n \in \mathrm{d}y\}\mathsf{P}\{X_{n+1} - X_n \geq -y\}) = \frac{\mathsf{P}\{X_{n+1} \geq 0, X_{n+1} - X_n \geq h(n)\}}{\mathsf{P}\{X_{1-\varepsilon} \geq 0\}}.$$

After applying (41) it is clear that $P_2 = o(\mathsf{P}\{X_n > 0\})$. Finally,

$$P_3 \leq \mathsf{P}\{X_n \geq h(n), X_{n+\varepsilon} - X_n < -h(n)\}$$
$$\leq \mathsf{P}\{X_n \geq 0\}\mathsf{P}\{X_\varepsilon < -h(n)\} = o(\mathsf{P}\{X_n \geq 0\}).$$

Now we should make use of the fact that if (39) holds for any fixed $\varepsilon \in (0,1)$, then, it holds uniformly in $\varepsilon \in (0,1)$. Consequently, (5) holds for $a(t) = \mathsf{P}\{X_t \geq 0\}/t$. The proof of (38) is similar. Finally, condition (6) for function $a(t)$ follows from the dominated convergence Theorem and the fact that for some constant C,

$$\int_1^{y-1} \frac{a(y-t)a(t)}{a(t)}\mathrm{d}t \leq C \sum_{k=1}^{[y-1]} \frac{a(k)a([y-1]-k)}{a[y-1]} < \infty.$$

The proof of Lemma A.2 is complete.

PROOF OF THEOREM 1.2. For Lévy processes the result follows directly from Theorem 1.1 and Lemma A.2. For random walks it can be proved along the lines of [13]. The only difference is that we should apply our Lemma A.1 instead of Lemma 4 in [13].

B Lévy processes

In this section we collect some facts from the theory of Lévy processes that we use in this paper.

This Lemma and its proof may be found in [33, Lemma 30.3].

Lemma B.1. *Let X_t be a Lévy process on $\mathbf{R}^{\mathrm{d}}$. For any $\varepsilon > 0$ there is $C = C(\varepsilon)$ such that, for any t,*

$$\mathsf{P}\{|X_t| > \varepsilon\} \leq Ct.$$

There are C_1, C_2 and C_3 such that, for any t ,

$$\mathsf{E}\{|X_t|^2; |X_t| \leq 1\} \leq C_1 t,$$
$$|\mathsf{E}\{X_t; |X_t| \leq 1\}| \leq C_2 t,$$
$$\mathsf{E}\{|X_t|; |X_t| \leq 1\} \leq C_3 t^{1/2}.$$

The next Theorem is a version of [33, theorem 25.3] adapted to our needs.

Theorem B.1. *Let X_t be a Lévy process on $\mathbf{R^d}$ with the Lévy measure ν. Then, X_t has a finite exponential moment if and only if $[\nu]_{|x|>1}$ has a finite exponential moment.*

Let F be an infinitely divisible law on $[0, +\infty)$. Its Laplace law can be expressed as ([33, theorem 30.1])

$$\int_0^\infty e^{-\lambda x} F(dx) = \exp\left\{ -\gamma\lambda - \int_0^\infty (1 - e^{-\lambda x})\nu(\mathrm{dx})) \right\},$$

where $\gamma \geq 0$ is a constant and ν is a Borel measure on $(0, \infty)$ for which $\mu \equiv \nu(1, \infty) < \infty$ and $\int_0^1 x\nu(\mathrm{dx}) < \infty$.

Theorem B.2. ([15, theorem 1]) *For F infinitely divisible on $[0, +\infty)$, the following assertions are equivalent:*

(i) $F \in \mathcal{S}$;
(ii) $\mu^{-1}\nu(1, x] \in \mathcal{S}$;
(iii) $\overline{F}(x) \sim \overline{\nu}(x)$.

We also need a density version of this Theorem.

Theorem B.3. *Let the infinitely divisible law F has a density f. Assume that there is x_0 such that ν has a density $g(x)$ for $x > x_0$. If $g(x)$ belongs to the class $\mathcal{S}d(\gamma)$, then*

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \int_0^\infty e^{\gamma x} f(x)\mathrm{dx}.$$

One can prove this Theorem exactly like [27, theorem 3.1] for a distribution function from $\mathcal{S}(\gamma)$. Corresponding properties of the class $\mathcal{S}d(\gamma)$ may be found in [21, Section 3].

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